1. Suppose $A = \begin{pmatrix} -3 & 0 & 0 \\ 6 & 1 & 0 \\ 0 & 4 & 2 \end{pmatrix}$.

   (a) Find A^{-1} using the Gauss-Jordan algorithm.
   (b) For a fixed $\mathbf{b} \in \mathbb{R}^3$, how many solutions could $A\mathbf{x} = \mathbf{b}$ have?
   (c) What is the column space of A?
   (d) What is the null space of A?
   (e) What is the rank of A?
   (f) Find a nontrivial dependence relation among the columns of A, or explain why none exists.
   (g) Find the inverse of A^2.
   (h) Suppose $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and $A(-2/9, -16, 7/5) = \mathbf{b}$. Find a solution to $AP\mathbf{x} = \mathbf{b}$.

Solution:

(a)

$$[A|I] = \left(\begin{array}{ccc|ccc} -3 & 0 & 0 & 1 & 0 & 0 \\ 6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

Multiply Row 1 by $-1/3$ to get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 6 & 1 & 0 & 0 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

Subtract 6(Row 1) from Row 2 to get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 4 & 2 & 0 & 0 & 1 \end{array}\right)$$

Subtract 4(Row 2) from Row 3 to get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & -8 & -4 & 1 \end{array}\right)$$

Multiply Row 3 by $1/2$ to get

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1/3 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -4 & -2 & 1/2 \end{array}\right)$$

Thus

$$A^{-1} = \begin{pmatrix} -1/3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -2 & 1/2 \end{pmatrix}.$$

Notice that the inverse of the lower triangular matrix A is also lower triangular. (Can you prove this is always true?)

(b) Since A is invertible, $A\mathbf{x} = \mathbf{b}$ always has one solution, $\mathbf{x} = A^{-1}\mathbf{b}$.

(c) Since A is invertible, the column space of A is all of $\mathbb{R}^3$.

(d) Since A is invertible, the null space of A is trivial, i.e. $N(A) = \{\mathbf{0}\}$.

(e) Since A is invertible, the rank of A is 3 (full row and column rank).

(f) Since A is invertible, the columns of A are linearly independent, so no nontrivial dependence relation among the columns of A exists.

(g) The inverse of A^2 is $(A^{-1})^2$, and

$$(A^{-1})^2 = \begin{pmatrix} -1/3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -2 & 1/2 \end{pmatrix} \begin{pmatrix} -1/3 & 0 & 0 \\ 2 & 1 & 0 \\ -4 & -2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/9 & 0 & 0 \\ 4/3 & 1 & 0 \\ -14/3 & -3 & 1/4 \end{pmatrix}$$

(h) Let

$$\mathbf{x} = P^T(-2/9, -16, 7/5) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -2/9 \\ -16 \\ 7/5 \end{pmatrix} = \begin{pmatrix} 7/5 \\ -2/9 \\ -16 \end{pmatrix}.$$

Then $AP\mathbf{x} = \mathbf{b}$, because

$$\begin{aligned} AP\mathbf{x} &= APP^T(-2/9, -16, 7/5) \\ &= A(PP^T)(-2/9, -16, 7/5) \\ &= AI(-2/9, -16, 7/5) \\ &= A(-2/9, -16, 7/5) \\ &= \mathbf{b}. \end{aligned}$$

2. Suppose A is a matrix where

$$A\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ -2 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ -10 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad A\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

(a) How many rows and columns does A have?

(b) What is the dimension of each of the four fundamental subspaces of A? Explain how you know.

(c) Find a basis for the column space of A.

(d) Find a basis for the null space of A.

(e) What is the rank of A?

(f) Find a solution to $A\mathbf{x} = (4, 4, -2, 1, 0)$ where the first component of $\mathbf{x}$ is 5.

(g) Find a solution to $A\mathbf{x} = (0, 0, 0, 0, 0)$ where the first component of $\mathbf{x}$ is -8.

(h) Describe all solutions to $A\mathbf{x} = (0, 4, -10, 0, 1)$

(i) Find a nontrivial dependence relation among the columns of A, or explain why none exists.

(j) For a fixed $\mathbf{b} \in \mathbb{R}^5$, how many solutions could $A\mathbf{x} = \mathbf{b}$ have?

(k) Suppose $\mathbf{b} = -2(4,4,-2,1,0) + 3(0,4,-10,0,1)$. Find a vector $\mathbf{x}$ where $A\mathbf{x} = \mathbf{b}$.

Solution:

(a) A has 5 rows and 3 columns.

(b) Since $\dim C(A) =$ the rank of A, the rank-nullity theorem implies $\dim C(A) + \dim N(A) = n = 3$. Since there are two independent vectors ($(4,4,-2,1,0)$ and $(0,4,-10,0,1)$) in the column space, $\dim C(A) \geq 2$. Since there is a nontrivial vector $(-2,1,0)$ in the null space, $\dim N(A) \geq 1$. Thus $\dim C(A) = 2$ and $\dim N(A) = 1$. Then $\dim C(A^T) = \dim C(A) = 2$, so using the rank-nullity theorem again, we conclude $\dim N(A^T) = m - r = 5 - 2 = 3$.

(c) The set $\{(4,4,-2,1,0),(0,4,-10,0,1)\}$ is a basis for the column space of A.

(d) The set $\{(-2,1,0)\}$ is a basis for the null space of A.

(e) The rank of A is 2.

(f) Since $(-2,1,0)$ is in the null space of A, and $A(1,0,1) = (4,4,-2,1,0)$, any vector of the form $\mathbf{x} = (1,0,1) + c(-2,1,0)$ will also be a solution to $A\mathbf{x} = (4,4,-2,1,0)$. Setting $c = -2$, we get the solution $\mathbf{x} = (5,-2,1)$.

(g) Since $(-2,1,0) \in N(A)$, so is any scalar multiple. Thus $\mathbf{x} = 4(-2,1,0) = (-8,4,0)$ is also in $N(A)$, and $A\mathbf{x} = (0,0,0,0,0)$.

(h) All solutions to $A\mathbf{x} = (0,4,-10,0,1)$ are of the form $\{(0,2,0) + c(-2,1,0) : c \in \mathbb{R}\}$.

(i) Since $A(-2,1,0) = \mathbf{0}$,

$$-2A_{*1} + A_{*2} = \mathbf{0}.$$

(j) Since A has neither full row rank or full column rank, for a fixed $\mathbf{b} \in \mathbb{R}^5$, $A\mathbf{x} = \mathbf{b}$ could have 0 or infinite solutions.

(k) Let $\mathbf{x} = -2(1,0,1) + 3(0,2,0) = (-2,6,-2)$. Then

$$A\mathbf{x} = A(-2\begin{pmatrix}1\\0\\1\end{pmatrix} + 3\begin{pmatrix}0\\2\\0\end{pmatrix}) = -2(A\begin{pmatrix}1\\0\\1\end{pmatrix}) + 3(A\begin{pmatrix}0\\2\\0\end{pmatrix}) = -2\begin{pmatrix}4\\4\\-2\\1\\0\end{pmatrix} + 3\begin{pmatrix}0\\4\\-10\\0\\1\end{pmatrix} = \mathbf{b}.$$

3. Suppose $\{\mathbf{v},\mathbf{w}\}$ is a linearly independent set of orthogonal vectors in $\mathbb{R}^4$, and $V = \text{span}\{\mathbf{v},\mathbf{w}\}$. Suppose A is a matrix where $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{0}$.

(a) State the definition of linear independence, i.e. Complete the sentence: "A set of vectors $\{\mathbf{v}_1,\ldots,\mathbf{v}_n\}$ is linearly independent if ...".

(b) State the definition of orthogonal vectors, i.e. Complete the sentence: "Vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if ...".

(c) Prove that the set $\{\mathbf{v}, \mathbf{v}-\mathbf{w}\}$ is linearly independent.

(d) Prove that the set $\{\mathbf{v}, \mathbf{v}-\mathbf{w}\}$ is not orthogonal.

(e) Which of the following (if any) must be true? Explain how you know.

- $V \subseteq N(A)$
- $N(A) \subseteq V$

(f) Which of the following (if any) must be true? Explain how you know.

- $V^\perp \subseteq C(A^T)$
- $C(A^T) \subseteq V^\perp$

Solution:

(a) A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is linearly independent if the only linear combination of these vectors equal to the zero vector is the trivial one: $0\mathbf{v}_1 + \cdots + 0\mathbf{v}_n = \mathbf{0}$.

(b) Vectors $\mathbf{v}$ and $\mathbf{w}$ are orthogonal if $\mathbf{v}^T\mathbf{w} = 0$.

(c) Suppose a and b are scalars where $a\mathbf{v}+b(\mathbf{v}-\mathbf{w}) = \mathbf{0}$. Since $a\mathbf{v}+b(\mathbf{v}-\mathbf{w}) = (a+b)\mathbf{v}-b\mathbf{w}$, this implies $(a+b)\mathbf{v}-b\mathbf{w} = \mathbf{0}$. But this is a linear combination of the vectors $\mathbf{v}$ and $\mathbf{w}$ equal to the zero vector, and since $\{\mathbf{v}, \mathbf{w}\}$ is linearly independent, this implies $a + b = 0$ and $-b = 0$. Thus $b = 0$ and $a = 0$ and the only linear combination of $\mathbf{v}$ and $\mathbf{v} - \mathbf{w}$ equal to the zero vector is the trivial one.

(d) Since $\mathbf{v}$ and $\mathbf{w}$ are orthogonal, $\mathbf{v}^T\mathbf{w} = 0$. Thus, using the linearity of the dot product, $\mathbf{v}^T(\mathbf{v}-\mathbf{w}) = \mathbf{v}^T\mathbf{v}-\mathbf{v}^T\mathbf{w} = \mathbf{v}^T\mathbf{v}$. Since $\mathbf{v}$ is not the zero vector (a zero vector cannot be part of a linearly independent set), $\mathbf{v}^T\mathbf{v} \neq 0$. Thus $\mathbf{v}^T(\mathbf{v} - \mathbf{w}) \neq 0$, and $\mathbf{v}$ and $\mathbf{v} - \mathbf{w}$ are not orthogonal.

(e) Since $\mathbf{v} \in N(A)$ and $\mathbf{w} \in N(A)$, any linear combination of these vectors is also in $N(A)$:

$$A(a\mathbf{v} + b\mathbf{w}) = a(A\mathbf{v}) + b(A\mathbf{w}) = a\mathbf{0} + b\mathbf{0} = \mathbf{0}.$$

Thus $V \subseteq N(A)$.

But there could be other vectors in $N(A)$ besides those in V, so it is not necessarily true that $N(A) \subseteq V$.

(f) Since $V \subseteq N(A)$, every vector orthogonal to $N(A)$ is also orthogonal to V. Since $C(A^T)$ is the orthogonal complement of $N(A)$ (containing all vectors orthogonal to $N(A)$), $C(A^T) \subseteq V^\perp$. However there could be additional vectors in $V^\perp$ that are not orthogonal to $N(A)$, so it is not necessarily true that $V^\perp \subseteq C(A^T)$.